Homework 8, due 11/11

1. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces, such that X is compact. Show that then Y is compact and f is surjective.

2. Let $f : X \to Y$ be a holomorphic map of Riemann surfaces, and let $x \in X$, $y = f(x) \in Y$. Show that we can find charts (U, ϕ) and (V, ψ) on X, Y containing x, y, such that
$$\psi \circ f \circ \phi^{-1}(z) = z^k,$$
for an integer k wherever the composition is defined.

3. Let $w_1, w_2 \in \mathbf{C}$ be given, such that $\mathrm{Im}(w_1/w_2) > 0$. Define the lattice $\Lambda = \{m_1 w_1 + m_2 w_2 \,:\, m_1, m_2 \in \mathbf{Z}\}$, and consider the complex torus $X = \mathbf{C}/\Lambda$.
   (a) Show that the 1-form dz on $\mathbf{C}$ defines a holomorphic 1-form α on X.
   (b) Consider the closed curve $\gamma(t) = tw_1$ for $t \in [0, 1]$ on X, and compute
   $$\int_\gamma \alpha.$$
   (c) Show that all holomorphic 1-forms on X are of the form $c\alpha$ for some $c \in \mathbf{C}$.

4. Consider the complex torus X from the previous problem. Define the function
$$g(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$
Show that g defines a meromorphic function on $\mathbf{C}$, which is periodic with respect to the lattice Λ, and so g defines a meromorphic function on X.

5. Let f be a non-constant meromorphic function on a compact Riemann surface X. Show that
$$\sum_p \mathrm{ord}_p(f) = 0,$$
where the sum is finite since $\mathrm{ord}_p(f)$ is only non-zero at poles or zeros of f.